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Diffraction radiation from a charge moving past an obstacle

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Abstract. This paper deals with the diffraction radiation generated by an electric charge, that moves at constant speed past a cylindrical obstacle. It is shown how the problem can be reduced to the problem of plane-wave diffraction when a suitable complex angle of incidence is introduced. Two alternative expressions for the radiated energy are obtained. The link between the two is furnished by the generalized form of the scattering cross section theorem. Both the moving line charge and the moving point charge are considered. As a special case, the wedge-like obstacle has been treated. Using Sommerfeld's type of integral representations, a solution of the relevant problem has been obtained. Numerical results pertaining to the total radiation loss of the point charge are presented.

1. Introduction

When a charged particle moves past a conducting structure it loses energy due to radiation, the so-called diffraction radiation. This phenomenon is of considerable importance in several domains in physics, in particular it occurs in particle accelerators. In a number of papers the diffraction radiation has been investigated, the most extensive investigation being by Bolotovskii and Voskresenskii (1966). In the present paper, the diffraction radiation from a charge moving past a cylindrical obstacle is considered. It will be shown that its calculation can be reduced to the problem of calculating the diffraction of a plane wave (with a complex angle of incidence) by the obstacle.

The usual procedure for obtaining the total radiation loss is to consider the energy radiated in the far field. In the present paper, an alternative expression for the total radiation loss is obtained by considering the mechanical work done to move the charge against the action of the diffracted field. Then, the physics of the interaction between the diffracted field and the moving charge becomes very distinct. The link between the two expressions obtained for the radiation loss is nothing but the well known scattering cross section theorem, when the latter is generalized to complex angles. Also, a reciprocity relation for the diffracted field is derived. First the two-dimensional problem of a line charge moving past the cylindrical obstacle is solved and after that the extensions to a moving point charge are outlined.

As a special case, the radiation from a line charge moving past a perfectly conducting wedge is considered. An exact solution is obtained by employing Sommerfeld's well known integral representations for plane-wave diffraction (Bowman *et al* 1969) even though the latter have to be extended to complex angles of incidence. Also, the problem of the point charge is considered. Some results that have been obtained by Gilinskii (1963) are improved and numerical results for the radiation loss of the point charge are presented.

2. Diffraction radiation from a line charge moving past a cylindrical obstacle

2.1. Formulation of the general problem

First we consider the case of a line charge with charge q per length in the y direction, moving in the x direction of a Cartesian coordinate system. The obstacle is cylindrical in the y direction and is assumed to be a perfect electrical conductor. The domain outside the obstacle is assumed to be a vacuum with permittivity ϵ_0 and permeability μ_0. The line charge moves with velocity $\boldsymbol{v} = v_0\boldsymbol{i}_x$ ($|v_0| < c_0$, c_0 = velocity of light in vacuum) along the trajectory $z = z_0$ = constant. For convenience we assume $z_0 > z_{\max}$ where $z_{\max}$ denotes the maximum attainable value of z at the obstacle. The y component of the magnetic-field vector is the fundamental unknown quantity and can be written (van den Berg 1973a) as

$$H_y(x, z, t) = \frac{1}{\pi}\mathrm{Re}\left(\int_0^\infty H_{y\omega}(x, z)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega\right) \tag{1}$$

with

$$H_{y\omega}(x, z) = V_0(V^{\mathrm{i}}(x, z) + V^{\mathrm{d}}(x, z)),$$

$$V_0 = \tfrac{1}{2}q\,\mathrm{sgn}(v_0)\exp[-k_0(c_0^2/v_0^2 - 1)^{1/2}\,z_0]$$

$$V^{\mathrm{i}}(x, z) = \exp(\mathrm{i}\alpha_0 x - \mathrm{i}\gamma_0 z) \qquad \text{when} \qquad -\infty < z < z_0,$$

in which

$$\alpha_0 = (\omega/v_0) = k_0(c_0/v_0), \qquad \gamma_0 = \mathrm{i}k_0(c_0^2/v_0^2 - 1)^{1/2}, \qquad k_0 = \omega(\epsilon_0\mu_0)^{1/2} = (\omega/c_0).$$

The incident field V^{i} pertains to the field of the moving line charge in the absence of the obstacle. The diffracted field V^{d} from the obstacle has to satisfy the source-free two-dimensional Helmholtz equation in the domain outside the obstacle. Further, the boundary condition $\partial V/\partial n = 0$ on the obstacle has to be satisfied, in which $V = V^{\mathrm{i}} + V^{\mathrm{d}}$ and $\partial/\partial n$ is the derivative along the outward normal to the boundary C of the obstacle.

For convenience, we further introduce the polar coordinates (ρ, ϕ) as

$$x = \rho\cos\phi, \qquad z = \rho\sin\phi, \qquad \text{with} \qquad 0 \leqslant \rho < \infty,\ 0 \leqslant \phi < 2\pi, \tag{2}$$

and the parameter ψ_0 through

$$\begin{aligned} |\alpha_0| &= k_0\cosh(\psi_0) = k_0\cos(\mathrm{i}\psi_0), \\ \gamma_0 &= \mathrm{i}k_0\sinh(\psi_0) = k_0\sin(\mathrm{i}\psi_0), \end{aligned} \tag{3}$$

with $0 < \psi_0 < \infty$. Then V^{i} can be written as

$$V^{\mathrm{i}}(\rho, \phi) = \exp[\mathrm{i}k_0\rho\cos(\phi - \phi_0)] \qquad \text{when} \qquad \pi \leqslant \phi \leqslant 2\pi, \tag{4}$$

with

$$\phi_0 = \begin{cases} -\mathrm{i}\psi_0 & \text{if } v_0 > 0 \\ \pi + \mathrm{i}\psi_0 & \text{if } v_0 < 0. \end{cases}$$

Hence, V^{i} can be regarded as an incident plane wave with a complex 'angle of incidence' ϕ_0. Consequently, when the known techniques in the theory of plane-wave diffraction are extended to complex angles of incidence, they can be used to attack the present problem of diffraction radiation.

2.2. *Radiation loss*

A representation for V^d can be obtained from the two-dimensional form of Green's theorem. The result is

$$V^d(x, z) = \int_C V(x', z') \frac{\partial G}{\partial n'} ds' \tag{5}$$

when (x, z) outside C, with $G = \frac{1}{4}iH_0^{(1)}[k_0(x-x')^2 + (z-z')^2]^{1/2}$.

Letting (x, z) approach C and invoking the boundary condition on C, we obtain an integral equation from which $V(x, z)$ on C can be solved. Once this has been done, the far field is obtained as

$$V^d(\rho, \phi) \simeq F_0(\phi)\left(\frac{i}{8\pi k_0\rho}\right)^{1/2} \exp(ik_0\rho) \qquad \text{as } k_0\rho \to \infty, \tag{6}$$

with

$$F_0(\phi) = \int_C V(x', z') \frac{\partial}{\partial n'} \exp[-ik_0\rho' \cos(\phi - \phi')] \, ds'. \tag{7}$$

From this, an expression for the total radiation loss is obtained by calculating the energy per length of the line charge radiated through a cylinder of unit length and large radius enclosing the obstacle. By using (1) and Maxwell's equations, we obtain

$$W = \int_{-\infty}^{\infty} dt \int_C \boldsymbol{n} \cdot (\boldsymbol{E}^d \times \boldsymbol{H}^d) \, ds = \frac{2}{\pi} \int_0^{\infty} W_\omega \, d\omega, \tag{8}$$

with

$$W_\omega = \tfrac{1}{2} \operatorname{Re}\left(\frac{V_0^2}{-i\omega\epsilon_0} \int_C V^d \frac{\partial V^{d*}}{\partial n} ds\right) = \frac{V_0^2}{16\pi\omega\epsilon_0} \int_0^{2\pi} |F_0(\phi)|^2 \, d\phi, \tag{9}$$

in which the asterisk denotes the complex conjugate. This expression, however, gives no direct insight in the mechanism of interaction of the field of the charge with the field diffracted by the obstacle. This situation greatly improves if, in (5), we employ the angular spectrum representation of the Hankel function (Morse and Feshbach 1953). Then we obtain for the diffracted field an expression of the form

$$V^d(x, z) = \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{B_0(\alpha)}{\gamma} \exp(i\alpha x + i\gamma z) \, d\alpha, \qquad z_{\max} < z < \infty, \tag{10}$$

in which

$$B_0(\alpha) = \int_C V(x', z') \frac{\partial}{\partial n'} \exp(-i\alpha x' - i\gamma z') \, ds', \tag{11}$$

with

$$\gamma = (k_0^2 - \alpha^2)^{1/2}, \qquad \operatorname{Re}(\gamma) \geq 0, \qquad \operatorname{Im}(\gamma) \geq 0.$$

We observe that (10) yields a representation in terms of a continuous plane-wave spectrum; its propagating waves ($|\alpha| < k_0$) represent the diffraction radiation in the domain $z_{\max} < z < \infty$. Now, the radiation loss per unit length of the line charge should be equal to the mechanical work W_{mech} per length of the line charge, done to move the line charge at a constant speed in the x direction against the action of the diffracted field from the obstacle. The latter is given by

$$W_{\text{mech}} = -\int_{-\infty}^{\infty} qv_0 E_x^{\text{d}}(v_0 t, z_0, t)\, \text{d}t, \tag{12}$$

in which, as $z_0 > z_{\max}$,

$$E_x^{\text{d}}(x, z, t) = \frac{-1}{4\pi^2} \text{Re}\left(\int_0^{\infty} \frac{V_0}{\text{i}\omega\epsilon_0} \text{d}\omega \int_{-\infty}^{\infty} B_0(\alpha) \exp[\text{i}(\alpha x + \gamma z - \omega t)]\, \text{d}\alpha\right). \tag{13}$$

The latter result has been derived with the use of (1), (10) and Maxwell's equations. Substituting (13) in (12) and interchanging the order of integration, we finally obtain

$$W_{\text{mech}} = \frac{2}{\pi} \int_0^{\infty} W_\omega\, \text{d}\omega \qquad \text{with} \qquad W_\omega = \tfrac{1}{2} \text{Re}\left(\frac{V_0^2}{\text{i}\omega\epsilon_0} B_0(\alpha_0)\right). \tag{14}$$

The quantity $(2/\pi) W_\omega\, \text{d}\omega$ can be interpreted as the differential radiation loss per length of the line charge caused by radiation with angular frequencies between ω and $\omega + \text{d}\omega$. It only depends on the amplitude $B_0(\alpha_0)$ of the spectral wave of (10) whose speed along the x direction is v_0. This is, as it should be, the only spectral wave that can interact with the incident wave generated by the particle.

Using (2) and (3) in the expression (11) for $B_0(\alpha_0)$ and comparing the result with the expression (7) for $F_0(\phi)$, we obtain

$$B_0(\alpha_0) = F_0(\phi_0^*), \qquad \phi_0^* = \begin{cases} \text{i}\psi_0 & \text{if } v_0 > 0, \\ \pi - \text{i}\psi_0 & \text{if } v_0 < 0, \end{cases} \tag{15}$$

in which the expression for the far-field amplitude is extended to a complex 'angle of observation'. As a matter of fact, the two expressions (9) and (14) for W_ω should yield the same result. That this indeed is the case, can be proved by observing that

$$\text{Im}\left(\int_C V^{\text{d}} \frac{\partial V^{\text{d}*}}{\partial n}\, \text{d}s\right) = -\text{Im}\left(\int_C V \frac{\partial V^{\text{i}*}}{\partial n}\, \text{d}s\right), \tag{16}$$

since $\partial V^{\text{d}}/\partial n = -\partial V^{\text{i}}/\partial n$ on C and $\text{Im}(\int_C V^{\text{i}} (\partial V^{\text{i}*}/\partial n)\, \text{d}s) = 0$. With either (1) and (11), or (4) and (7) we further have

$$\int_C V \frac{\partial V^{\text{i}*}}{\partial n}\, \text{d}s = B_0(\alpha_0) = F_0(\phi_0^*) \tag{17}$$

and hence

$$\frac{1}{8\pi} \int_0^{2\pi} |F_0(\phi)|^2\, \text{d}\phi = \text{Im}(F_0(\phi_0^*)). \tag{18}$$

In this context, relation (18) is the well known scattering cross section theorem (Jones 1955, de Hoop 1959) in the theory of the plane-wave diffraction, but now extended to complex angles.

Along similar lines, a reciprocity relation for the diffracted field can be derived. Let V_a and V_b be the fundamental field functions to line charges moving with velocities v_a and v_b in the x direction, respectively. Let $\alpha_{a,b}$, $\psi_{a,b}$, $\phi_{a,b}$, $F_{a,b}$ and $B_{a,b}$ denote the relevant quantities defined by (1), (3), (4), (7) and (11). Following the method outlined by de Hoop (1960) in the plane-wave diffraction theory, we then obtain from Green's reciprocity theorem

$$\int_C V_a \frac{\partial V_b^{\mathrm{i}}}{\partial n}\,\mathrm{d}s = \int_C V_b \frac{\partial V_a^{\mathrm{i}}}{\partial n}\,\mathrm{d}s. \tag{19}$$

With either (1) and (11), or (4) and (7) we now observe that

$$B_a(-\alpha_b) = B_b(-\alpha_a), \qquad \text{or} \qquad F_a(\pi+\phi_b) = F_b(\pi+\phi_a), \tag{20}$$

which is the desired reciprocity relation. In the special case $\alpha_a = -\alpha_b = \alpha_0$, we have

$$B_a(\alpha_0) = B_b(-\alpha_0), \tag{21}$$

i.e. if we replace v_0 by $-v_0$ and keep z_0 the same; it follows from (14) and (21) that the radiation loss is the same.

The results derived so far apply to cylindrical obstacles of bounded cross section, but it can be shown that the same results apply to cylindrical obstacles of unbounded cross section, e.g. wedge-shaped obstacles.

2.3. *The special case of a wedge-shaped obstacle*

In the special case of a wedge-like obstacle (figure 1) an exact solution can be obtained by employing Sommerfeld's well known integral representations for plane-wave diffraction by a wedge (Bowman *et al* 1969). Assuming that $z_{\max} = 0$, $v_0 > 0$ and

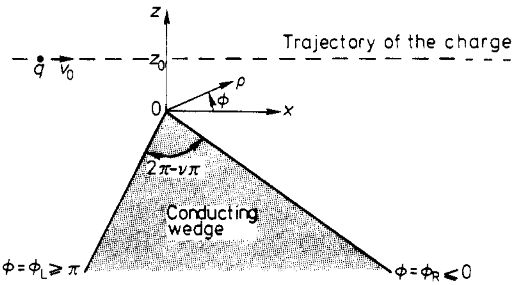

Figure 1. Cross section of the wedge configuration; $\nu = (\phi_L - \phi_R)/\pi$, $1 \leq \nu \leq 2$.

introducing a complex angle of incidence $\phi_0 = -\mathrm{i}\psi_0$, the far-field amplitude is directly obtained as

$$F_0(\phi) = \frac{2}{\nu}\left(\frac{\sin(\pi/\nu)}{\cos(\pi/\nu) - \cos[(\phi - \pi + \mathrm{i}\psi_0)/\nu]} + \frac{\sin(\pi/\nu)}{\cos(\pi/\nu) - \cos[(\phi + \pi - 2\phi_L - \mathrm{i}\psi_0)/\nu]}\right). \tag{22}$$

in which $\nu = (\phi_L - \phi_R)/\pi$. In the special case of a half-plane ($\nu = 2$), expression (22) agrees with the one given by Bolotovskii and Voskresenskii (1966), obtained by using the Wiener–Hopf technique.

The spectral density W_ω of the radiation loss can now be written as

$$W_\omega = \frac{V_0^2}{16\pi\omega\epsilon_0}\int_{\phi_R}^{\phi_L} |F_0(\phi)|^2\,\mathrm{d}\phi = \frac{V_0^2}{2\omega\epsilon_0}\,\mathrm{Im}(F_0(\mathrm{i}\psi_0)). \tag{23}$$

The integration with respect to ϕ in the first expression for W_ω is rather difficult to carry out, therefore we take the second expression and directly obtain

$$W_\omega = \frac{V_0^2}{2\omega\epsilon_0}\frac{2}{\nu}\,\mathrm{Im}\left(\frac{\sin(\pi/\nu)}{\cos(\pi/\nu)-\cos[(2\mathrm{i}\psi_0-\pi)/\nu]}\right). \tag{24}$$

For a half-plane, we have

$$W_\omega = \frac{V_0^2}{2\omega\epsilon_0}\,\mathrm{Im}\left(\frac{-1}{\sin(\mathrm{i}\psi_0)}\right) = \frac{q^2\exp[-2k_0(c_0^2/v_0^2-1)^{1/2}z_0]}{8\omega\epsilon_0(c_0^2/v_0^2-1)^{1/2}}. \tag{25}$$

Expression (25) is in complete agreement with the one obtained by Bolotovskii and Voskresenskii (1966), when the integration in (23) with respect to ϕ is carried out. The result in (24) is independent of the orientation of the wedge; it only depends on the included angle of the wedge.

To find the total radiation loss W, we must integrate with respect to ω. Obviously, the integral of W_ω diverges logarithmically at low frequencies. This is due to the fact that the energy contained in the field of the moving line charge diverges logarithmically, the field varying inversely proportional to the distance from the trajectory of the moving line charge.

For small velocities of the line charge, we have $v_0 \ll c_0$ (i.e. ψ_0 is large), we then obtain from (22)

$$|F_0(\phi)| \simeq 8\left(\frac{v_0}{2c_0}\right)^{1/\nu}\frac{\sin(\pi/\nu)}{\nu}\,|\cos[(\phi-\phi_\mathrm{L})/\nu]| \qquad \text{when } v_0 \ll c_0, \tag{26}$$

and

$$W_\omega \simeq \frac{q^2\exp[-2k_0(c_0/v_0)z_0]}{2\omega\epsilon_0}\left(\frac{v_0}{2c_0}\right)^{(2/\nu)}\frac{\sin^2(\pi/\nu)}{\nu} \qquad \text{when } v_0 \ll c_0. \tag{27}$$

For large velocities of the line charge, we have $v_0 \to c_0$ (i.e. $\psi_0 \to 0$). Then $|F_0(\phi)|$ has maxima along the directions $\phi = 0$ and $\phi = 2\phi_\mathrm{L} - 2\pi$. These angles correspond to radiation in the direction of motion of the line charge and radiation in the image direction with respect to the left-side $\phi = \phi_\mathrm{L}$ of the wedge. The angular width of these maxima is of the order $(c_0^2/v_0^2-1)^{1/2}$ as $v_0 \to c_0$. Further, it can be shown that

$$W_\omega \simeq \frac{q^2\exp[-2k_0(c_0^2/v_0^2-1)^{1/2}z_0]}{8\omega\epsilon_0(c_0^2/v_0^2-1)^{1/2}} \qquad \text{when } v_0 \to c_0, \tag{28}$$

provided that $(c_0^2/v_0^2-1)^{1/2} \ll \nu\sin(\pi/\nu)$. When $\nu = 1$, in the nature of things, the radiation loss equals zero. From (25) and (28) we observe that at large velocities of the line charge the radiation loss becomes independent of the included angle of the wedge and approaches the one due to a half-plane.

3. Diffraction radiation from a point charge moving past a cylindrical obstacle

3.1. Formulation of the general problem

In this section, we consider the case of an electric point charge q moving with velocity $\boldsymbol{v} = v_0\boldsymbol{i}_x$ along the trajectory $y = 0$, $z = z_0 = \text{constant} > z_{\max}$. The y components of the electric field vector and the magnetic field vector are the fundamental unknown quantities and can be written as (van den Berg 1973b)

$$\begin{Bmatrix} E_y(x, y, z, t) \\ H_y(x, y, z, t) \end{Bmatrix} = \frac{1}{2\pi^2} \mathrm{Re}\left(\int_0^\infty \mathrm{d}\omega \int_{-\infty}^{\infty} \begin{Bmatrix} E_{y\omega}(x, z; \beta) \\ H_{y\omega}(x, z; \beta) \end{Bmatrix} \exp(\mathrm{i}\beta y - \mathrm{i}\omega t)\, \mathrm{d}\beta \right), \qquad \begin{matrix} (29) \\ (30) \end{matrix}$$

with

$$E_{y\omega}(x, z; \beta) = U_\beta(U^{\mathrm{i}}(x, z; \beta) + U^{\mathrm{d}}(x, z; \beta)),$$

$$H_{y\omega}(x, z; \beta) = V_\beta(V^{\mathrm{i}}(x, z; \beta) + V^{\mathrm{d}}(x, z; \beta)),$$

$$U_\beta = \tfrac{1}{2} q \left(\frac{\mu_0}{\epsilon_0} \right)^{1/2} \frac{\beta}{k_0} \frac{|\alpha_0|}{\gamma_\beta} \exp(\mathrm{i}\gamma_\beta z_0),$$

$$V_\beta = \tfrac{1}{2} q \,\mathrm{sgn}(v_0) \exp(\mathrm{i}\gamma_\beta z_0),$$

$$U^{\mathrm{i}}(x, z; \beta) = V^{\mathrm{i}}(x, z; \beta) = \exp(\mathrm{i}\alpha_0 x - \mathrm{i}\gamma_\beta z) \qquad \text{when } -\infty < z < z_0,$$

in which

$$\gamma_\beta = \mathrm{i}[k_0^2(c_0^2/v_0^2 - 1) + \beta^2]^{1/2}.$$

U^{d} and V^{d} are now solutions of the source-free two-dimensional Helmholtz equation in the domain outside the obstacle, but with wavenumber

$$k_\beta = (k_0^2 - \beta^2)^{1/2}, \qquad \mathrm{Re}(k_\beta) \geq 0, \qquad \mathrm{Im}(k_\beta) \geq 0. \tag{31}$$

Further, the boundary conditions $U = 0$ and $\partial V/\partial n = 0$ on the boundary C of the obstacle have to be satisfied, in which $U = U^{\mathrm{i}} + U^{\mathrm{d}}$ and $V = V^{\mathrm{i}} + V^{\mathrm{d}}$.

For convenience, we can again introduce the polar coordinates (ρ, ϕ) as defined in (2) and a parameter ψ_β in the range $|\beta| < k_0$, through

$$\begin{aligned} |\alpha_0| &= k_\beta \cosh(\psi_\beta) = k_\beta \cos(\mathrm{i}\psi_\beta), \\ \gamma_\beta &= \mathrm{i}k_\beta \sinh(\psi_\beta) = k_\beta \sin(\mathrm{i}\psi_\beta), \end{aligned} \qquad 0 < \psi_\beta < \infty \quad \text{when} \quad |\beta| < k_0. \tag{32}$$

Then, the incident field functions can be written as

$$U^{\mathrm{i}}(\rho, \phi; \beta) = V^{\mathrm{i}}(\rho, \phi; \beta) = \exp[\mathrm{i}k_\beta \rho \cos(\phi - \phi_\beta)], \qquad \pi \leq \phi \leq 2\pi,\ |\beta| < k_0, \tag{33}$$

in which

$$\phi_\beta = \begin{cases} -\mathrm{i}\psi_\beta & \text{if } v_0 > 0, \\ \pi + \mathrm{i}\psi_\beta & \text{if } v_0 < 0. \end{cases}$$

Hence, (33) can be regarded as an incident plane wave with wavenumber k_β and complex angle of incidence ϕ_β.

3.2. Radiation loss

In order to investigate the radiated energy, we now start with the alternative expressions for U^{d} and V^{d} namely the far-field representations

$$\begin{Bmatrix} U^{\mathrm{d}}(\rho, \phi; \beta) \\ V^{\mathrm{d}}(\rho, \phi; \beta) \end{Bmatrix} \simeq \begin{Bmatrix} E_\beta(\phi) \\ F_\beta(\phi) \end{Bmatrix} (\mathrm{i}/8\pi k_\beta \rho)^{1/2} \exp(\mathrm{i}k_\beta \rho) \qquad \text{as } |k_\beta \rho| \to \infty, \tag{34}$$

in which

$$E_\beta(\phi) = -\int_C \frac{\partial U}{\partial n'} \exp[-\mathrm{i}k_\beta \rho' \cos(\phi - \phi')]\,\mathrm{d}s',$$
$$F_\beta(\phi) = \int_C V\frac{\partial}{\partial n'} \exp[-\mathrm{i}k_\beta \rho' \cos(\phi - \phi')]\,\mathrm{d}s', \tag{35}$$

and the angular-spectrum representations

$$\begin{Bmatrix} U^{\mathrm{d}}(x,z;\beta) \\ V^{\mathrm{d}}(x,z;\beta) \end{Bmatrix} = \frac{\mathrm{i}}{4\pi}\int_{-\infty}^{\infty} \begin{Bmatrix} A_\beta(\alpha) \\ B_\beta(\alpha) \end{Bmatrix} \gamma^{-1} \exp(\mathrm{i}\alpha x + \mathrm{i}\gamma z)\,\mathrm{d}\alpha \qquad \text{when } z_{\max} < z < \infty, \tag{36}$$

in which

$$A_\beta(\alpha) = -\int_C \frac{\partial U}{\partial n'} \exp(-\mathrm{i}\alpha x' - \mathrm{i}\gamma z')\,\mathrm{d}s',$$
$$B_\beta(\alpha) = \int_C V\frac{\partial}{\partial n'} \exp(-\mathrm{i}\alpha x' - \mathrm{i}\gamma z')\,\mathrm{d}s'. \tag{37}$$

In (36) and (37), we have $\gamma = (k_\beta^2 - \alpha^2)^{1/2}$, $\mathrm{Re}(\gamma) \geqslant 0$, $\mathrm{Im}(\gamma) \geqslant 0$, which is not to be mixed up with the definition of γ in (11). In the same way as in the case of the line charge, we now obtain two alternative expressions for the radiation loss of the point charge

$$W = \frac{1}{\pi^2}\int_0^\infty \mathrm{d}\omega \int_{-\infty}^{\infty} W_\omega(\beta)\,\mathrm{d}\beta, \tag{38}$$

with either

$$W_\omega(\beta) = \tfrac{1}{2}\mathrm{Re}\left(\frac{\mathrm{i}\omega\epsilon_0}{k_\beta^2}\int_C U^{\mathrm{d}}\frac{\partial U^{\mathrm{d}*}}{\partial n}\,\mathrm{d}s + \frac{\mathrm{i}\omega\mu_0}{k_\beta^2}\int_C V^{\mathrm{d}}\frac{\partial V^{\mathrm{d}*}}{\partial n}\,\mathrm{d}s\right) \tag{39}$$

$$= \begin{cases} \dfrac{\omega}{16\pi k_\beta^2}\left(\epsilon_0 |U_\beta|^2 \displaystyle\int_0^{2\pi} |E_\beta(\phi)|^2\,\mathrm{d}\phi + \mu_0 V_\beta^2 \int_0^{2\pi} |F_\beta(\phi)|^2\,\mathrm{d}\phi\right) & \text{when } |\beta| < k_0, \\ 0 & \text{when } |\beta| \geqslant k_0, \end{cases}$$

or

$$W_\omega(\beta) = \frac{\omega}{2k_\beta^2}[\epsilon_0 |U_\beta|^2\,\mathrm{Im}(A_\beta(\alpha_0)) + \mu_0 V_\beta^2\,\mathrm{Im}(B_\beta(\alpha_0))]. \tag{40}$$

Obviously, the radiation loss only depends on the amplitudes $A_\beta(\alpha_0)$ and $B_\beta(\alpha_0)$ of those spectral waves in (36) whose speed in the x direction equals v_0. From (39), it follows that the integration with respect to β in (38) reduces to $|\beta| < k_0$. This criterion arises entirely from the fact that the whole field is then evanescent. The two expressions (39) and (40) should yield the same result, hence

$$\frac{1}{8\pi}\int_0^{2\pi} \begin{Bmatrix} |E_\beta(\phi)|^2\,\mathrm{d}\phi \\ 0 \end{Bmatrix} = \mathrm{Im}(A_\beta(\alpha_0)) = \begin{cases} \mathrm{Im}(E_\beta(\phi_\beta^*)) & \text{when } |\beta| < k_0, \\ 0 & \text{when } |\beta| \geqslant k_0, \end{cases}$$
$$\frac{1}{8\pi}\int_0^{2\pi} \begin{Bmatrix} |F_\beta(\phi)|^2\,\mathrm{d}\phi \\ 0 \end{Bmatrix} = \mathrm{Im}(B_\beta(\alpha_0)) = \begin{cases} \mathrm{Im}(F_\beta(\phi_\beta^*)) & \text{when } |\beta| < k_0, \\ 0 & \text{when } |\beta| \geqslant k_0. \end{cases} \tag{41}$$

These results can be proved directly in the same manner as in § 2.2 by using Green's theorem. Similarly, reciprocity relations like (20) and (21) can be derived. Again, reversing the direction of motion of the point charge does not change the radiation loss.

3.3. *The special case of a wedge-like obstacle*

The corresponding wedge-problem for a point charge has been considered by Gilinskii (1963), but his method of analysis is very cumbersome, since a Hertzian vector is introduced and integral representations of Malyuzhinets and Tuzhilin (1963) for a Hertzian vector are used. No final results with respect to the radiation loss are presented. Therefore, a reconsideration of the problem seems appropriate. In this special case of a wedge-like obstacle (figure 1), we again employ Sommerfeld's integral representations for plane-wave diffraction by a wedge. Assuming that $z_{\max}=0$, $v_0>0$ and introducing a complex angle of incidence $\phi_\beta=-\mathrm{i}\psi_\beta$, the far-field amplitudes are directly obtained as (Bowman *et al* 1969)

$$\begin{Bmatrix} E_\beta(\phi) \\ F_\beta(\phi) \end{Bmatrix} = \frac{2}{\nu}\left(\frac{\sin(\pi/\nu)}{\cos(\pi/\nu)-\cos[(\phi-\pi+\mathrm{i}\psi_\beta)/\nu]} \mp \frac{\sin(\pi/\nu)}{\cos(\pi/\nu)-\cos[(\phi+\pi-2\phi_{\mathrm{L}}-\mathrm{i}\psi_\beta)/\nu]}\right), \qquad |\beta|<k_0. \tag{42}$$

The radiation loss of the point charge can then be obtained as

$$W=\frac{1}{\pi^2}\int_0^\infty \mathrm{d}\omega \int_{-k_0}^{k_0} W_\omega(\beta)\,\mathrm{d}\beta, \tag{43}$$

with

$$\begin{aligned} W_\omega(\beta) &= \frac{\omega}{16\pi k_\beta^2}\left(\epsilon_0|U_\beta|^2\int_{\phi_{\mathrm{R}}}^{\phi_{\mathrm{L}}}|E_\beta(\phi)|^2\,\mathrm{d}\phi+\mu_0V_\beta^2\int_{\phi_{\mathrm{R}}}^{\phi_{\mathrm{L}}}|F_\beta(\phi)|^2\,\mathrm{d}\phi\right) \\ &= \frac{\omega}{2k_\beta^2}[\epsilon_0|U_\beta|^2\,\mathrm{Im}(E_\beta(\mathrm{i}\psi_\beta))+\mu_0V_\beta^2\,\mathrm{Im}(F_\beta(\mathrm{i}\psi_\beta))]. \end{aligned}$$

We now introduce the angle θ as

$$\beta=k_0\cos\theta \qquad \text{with } 0<\theta<\pi. \tag{44}$$

Taking into account the y dependence $\exp(\mathrm{i}\beta y)$ of each spatial spectral component, it is obvious that ϕ and θ are the angles of emergence of a radiating wave. Using these, (43) can be rewritten as

$$W=\frac{1}{\pi^2}\int_0^\infty \mathrm{d}\omega\int_0^\pi \sin(\theta)\,\mathrm{d}\theta\int_{\phi_{\mathrm{R}}}^{\phi_{\mathrm{L}}} W_\omega(\phi,\theta)\,\mathrm{d}\phi, \tag{45}$$

with

$$\begin{aligned} W_\omega(\phi,\theta) &= \frac{q^2}{64\pi\epsilon_0c_0}\exp\left[-2k_0\left(\frac{c_0^2}{v_0^2}-\sin^2\theta\right)^{1/2}z_0\right] \\ &\quad\times\left(\cos^2\theta\,\frac{c_0^2/v_0^2}{c_0^2/v_0^2-\sin^2\theta}\,|E_\beta(\phi)|^2+|F_\beta(\phi)|^2\right)(\sin\theta)^{-2} \qquad \text{as } \beta=k_0\cos\theta. \end{aligned} \tag{46}$$

Hence $(1/\pi^2) W_\omega(\phi, \theta)\, d\omega\, d\Omega$ is the differential radiation loss in the range of angular frequencies between ω and $\omega + d\omega$, radiated into the solid angle $d\Omega = \sin(\theta)\, d\theta\, d\phi$. In order to obtain the total radiation loss of the point, we prefer (43) with the second expression for $W_\omega(\beta)$. Since $\mathrm{Im}(E_\beta(i\psi_\beta)) = \mathrm{Im}(F_\beta(i\psi_\beta))$, we arrive after performing the integration with respect to ω, at

$$W = \frac{q^2}{4\nu\pi^2\epsilon_0 z_0} \int_0^{\frac{1}{2}\pi} \mathrm{Im}\left(\frac{\sin(\pi/\nu)}{\cos(\pi/\nu) - \cos[(2i\psi_\beta - \pi)/\nu]}\right) \times \frac{\cos^2\theta[(c_0^2/v_0^2)/(c_0^2/v_0^2 - \sin^2\theta)] + 1}{\sin(\theta)(c_0^2/v_0^2 - \sin^2\theta)^{1/2}}\, d\theta \tag{47}$$

in which

$$\cosh(\psi_\beta) = \frac{c_0}{v_0 \sin\theta}.$$

This result is independent of the orientation of the wedge, depending only on the included angle of the wedge. In the special case of a half-plane ($\nu = 2$) the integration in (47) can be carried out analytically

$$W_{\nu=2} = \frac{3q^2}{32\pi\epsilon_0 z_0 (c_0/v_0)(c_0^2/v_0^2 - 1)^{1/2}}. \tag{48}$$

Expression (48) is in agreement with the one obtained by Kazantsev and Surdutovich (1963) who used the Wiener–Hopf technique. For arbitrary values of ν, the integration in (47) is carried out numerically and the results are shown in figure 2. We observe that the total radiation loss W for a wedge is always smaller than the one pertaining to a half-plane. Further, we observe that for a very flat wedge-shaped obstacle, the losses are very small except in the case of ultra-relativistic velocity, where a very sharp increase takes place with increasing velocity.

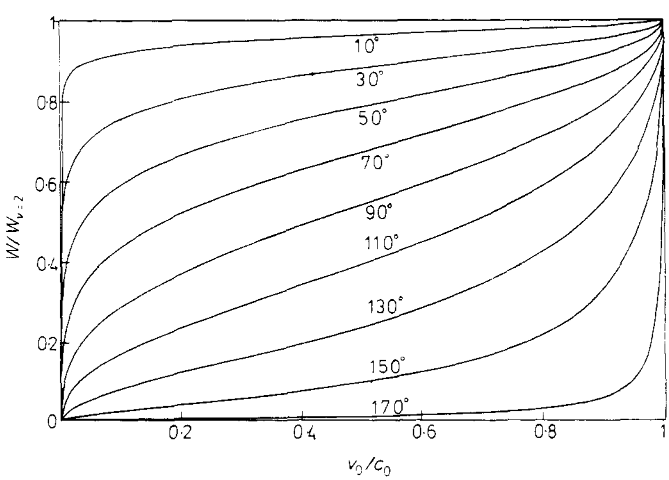

Figure 2. The ratio of the radiation loss W of a point charge moving past a wedge and the relevant value $W_{\nu=2}$ for a half-plane. The figure near a curve denotes the included angle (in degrees) of the wedge under consideration. The orientation of the wedge is arbitrary.

For small velocities of the point charge, we have $v_0 \ll c_0$ (i.e. ψ_β is large). Then we obtain

$$W_\omega(\phi, \theta) \simeq \frac{q^2}{\pi\epsilon_0 c_0} \exp\left(-2k_0 \frac{c_0}{v_0} z_0\right)\left(\frac{v_0}{2c_0}\right)^{2/\nu} \frac{\sin^2(\pi/\nu)}{\nu^2}(\sin\theta)^{-2+(2/\nu)}$$

$$\times\{\cos^2(\theta)\sin^2[(\phi-\phi_\mathrm{L})/\nu]+\cos^2[(\phi-\phi_\mathrm{L})/\nu]\} \tag{49}$$

and

$$W \simeq \frac{q^2}{\pi^{3/2}\epsilon_0 z_0}\left(\frac{v_0}{2c_0}\right)^{1+(2/\nu)} \frac{\sin^2(\pi/\nu)}{\nu} \frac{\Gamma(1/\nu)(\nu+1)}{\Gamma(\frac{1}{2}+1/\nu)(\nu+2)} \qquad \text{when } v_0 \ll c_0. \tag{50}$$

in which $\Gamma(\cdot)$ denotes the gamma function.

For large velocities of the point charge, we have $v_0 \to c_0$. Then $W_\omega(\phi, \theta)$ has the maxima along the directions $\phi = 0$, $\theta = \frac{1}{2}\pi$ and $\phi = 2\phi_\mathrm{L} - 2\pi$, $\theta = \frac{1}{2}\pi$. These angles correspond to radiation in the direction of motion of the point charge and radiation in the image direction with respect to the left-side plane $\phi = \phi_\mathrm{L}$ of the wedge. The angular width of these maxima is of order $[c_0^2/(v_0 \sin\theta)^2 - 1]^{1/2}$ when $v_0 \to c_0$. Further it can be shown that

$$W \to W_{\nu=2} \qquad \text{as } v_0 \to c_0,\ \nu \neq 1, \tag{51}$$

as we have already observed in figure 2.

4. Conclusions

In the present paper, a method has been presented of how problems in diffraction radiation can be handled rigorously. It is shown that they can be formulated in terms of plane-wave diffraction with complex angles of incidence. Further, it is shown how the radiated energy can directly be obtained from the far-field amplitude of the diffracted field, 'observed' at a specified complex angle. The result is extremely useful, since the sometimes difficult integration of the angular distribution of the far-field amplitude can be avoided. Further, the mechanism of interaction of the diffracted field with the moving charge is made more distinct.

As a special case the consequences of a wedge-like obstacle are considered. The radiation loss is always independent of the orientation of the wedge and always less than the one pertaining to a semi-infinite screen. In the case of ultra-relativistic velocities and when the wedge is very flat (say with an included angle of 170–180°), a very sharp increase of the radiation with increasing velocity takes place.

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